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Dirac Equation in Robertson–Walker Metric

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We recast the Dirac relativistic equation within the theoretical framework of the Robertson–Walker metric, using spatial hypersurfaces that are essentially curved, and hence more general, as compared to the flat ones employed by Barut and Duru.

Given the tremendous success with which the celebrated Dirac relativistic equation has been credited, it has been overwhelmingly tempting to extend its range of validity to such remote areas as astrophysics and cosmology, where gravity is believed to play a dominant role in determining the behavior of spin-1/2 particles (Brill and Wheeler, 1957; Isham and Nelson, 1974; Chimento and Mollerach, 1986; Audretsch and Schäfer, 1978; Dehnen and Schäfer, 1980). Earlier attempts in this direction have had their share of intrigues and limitations; the latest one being by Barut and Duru (1987), who based their formulation of the Dirac equation in the Robertson–Walker (RW) metric on the assumption of spatially "flat" hypersurfaces, ostensibly for reasons of simplicity. A full-fledged description, on the other hand, would certainly require consideration of a treatment based on hypersurfaces that are explicitly spatially "curved" instead of the flat ones. This is precisely the chief motivation for the present investigation, where we develop the formulation of the Dirac equation within the general theoretical framework of the full RW metric, which is undoubtedly compatible with a realistic situation governing the dynamics of spin-1/2 particles in expanding space-times.

For purposes of elucidation, we delineate only the salient features of the formulation. The Dirac equation in the context of a spatially curved RW metric for a particle of mass m can be expressed as (Parashar, 1990)

$$[\gamma^{\mu}(x)\partial_{\mu} - \gamma^{\mu}(x)\Gamma_{\mu}(x) + im]\psi = 0 \tag{1}$$

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where $\gamma^{\mu}(x)$ are Dirac matrices depending on curved space and $\Gamma_{\mu}(x)$ are the spin connections, which can be evaluated in terms of the affine connections for the full RW metric,

$$ds^2 = dt^2 - a^2(t)[\eta^{-2}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \tag{2}$$

where $a(t)$ is some unknown function of the cosmic standard time t, and η is defined as $\eta^2 = 1 - kr^2$, k being a constant whose values can be chosen to be 0, +1, or −1, in certain appropriate units for r.

The anticommutation relation for the curved Dirac matrices can be written down as

$$\{\gamma^{\mu}(x), \gamma^{\nu}(x)\} = 2g^{\mu\nu}(x) \tag{3}$$

where $g^{\mu\nu}(x)$ is a nondiagonal metric tensor whose components in terms of the contents of the RW metric (expressed in Cartesian coordinates for direct application) have the following structures:

$$g^{00} = 1, \qquad g^{0i} = g^{i0} = 0, \qquad g^{ij} = -\frac{1}{a^2}(\delta_{ij} - kx_i x_j) \tag{4}$$

Similarly, the components of $g_{\mu\nu}(x)$ can be computed from those of $g^{\mu\nu}(x)$ by invoking the relation $g^{\mu\nu} = (-)^{\mu+\nu} M_{\nu\mu}/g$, $g = \det g_{\mu\nu} = -\eta^{-2}a^6$. The corresponding structures are

$$g_{00} = 1, \qquad g_{0i} = g_{i0} = 0, \qquad g_{ij} = -\eta^{-2}a^2(\eta^2\delta_{ij} + kx_i x_j) \tag{5}$$

The concept of vierbein is exploited to obtain a relationship between the curved Dirac matrices (having an argument and characterized by upper indices) and the standard flat ones (without argument and identified by lower indices) with the result

$$\gamma^{\mu}(x) = U^{\mu\nu}(x)\gamma_{\nu} \tag{6}$$

$$U^{00} = 1, \qquad U^{0i} = U^{i0} = 0, \qquad U^{ij} = \frac{1}{ar^2}(\xi x_i x_j - r^2\delta_{ij}) \tag{7}$$

Here $\xi = 1 - \eta$ is a measure of the effect of the curvature of space.

We now calculate the (space-dependent) spin connections $\Gamma_{\mu}(x)$, which satisfy the equation (Barut and Duru, 1987)

$$[\Gamma_{\mu}(x), \gamma^{\nu}(x)] = \partial_{\mu}\gamma^{\nu}(x) + \Gamma^{\nu}_{\mu\lambda}\gamma^{\lambda}(x) \tag{8}$$

where $\Gamma^{\nu}_{\mu\lambda}(x)$ are the corresponding affine connections, which can be computed from the relations (Weinberg, 1971)

$$\Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij}, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j, \qquad \Gamma^i_{jk} = \tfrac{1}{2}(\tilde{g}^{-1})^{il}[\partial_k \tilde{g}_{lj} + \partial_j \tilde{g}_{lk} - \partial_l \tilde{g}_{jk}] \quad (9)$$

where $\tilde{g}_{ij}$ are defined by $g_{ij} = a^2 \tilde{g}_{ij}$. The nonvanishing elements of the affine connections $\Gamma^{\nu}_{\mu\lambda}(x)$ for the RW metric are given by

$$\Gamma^0_{ij} = a\dot{a}\eta^{-2}(\eta^2\delta_{ij} + kx_ix_j), \qquad \Gamma^l_{0l} = \Gamma^l_{l0} = \frac{\dot{a}}{a}$$
$$\Gamma^l_{ij} = \eta^{-2}kx_l(\eta^2\delta_{ij} + kx_ix_j) \quad (10)$$

The spin connections $\Gamma_\mu(x)$ admit of parametrization in terms of the bilinear combinations of the flat Dirac matrices. The essential ingredients are assimilated in the relation

$$\Gamma_\mu(x) = V^{\nu\lambda}_\mu(x)\gamma_\nu\gamma_\lambda, \qquad \nu < \lambda \quad (11)$$

where the entire proliferation of parameters has been worked out on the basis of the above prescriptions, resulting in the following structures $(\Lambda = 1 - \eta^{-1})$:

$$V^{\nu\lambda}_0(x) = 0, \qquad V^{jl}_i(x) = 0, \qquad (i \neq j \neq l)$$
$$V^{0j}_i(x) = V^{0i}_j(x) = -\frac{\dot{a}}{2r^2}(\Lambda x_ix_j - r^2\delta_{ij}) \quad (12)$$
$$V^{12}_1 = -\frac{\xi x_2}{2r^2} = -V^{23}_3, \qquad V^{13}_1 = -\frac{\xi x_3}{2r^2} = V^{23}_2, \qquad V^{12}_2 = \frac{\xi x_1}{2r^2} = V^{13}_3$$

Equations (6) and (7) comprising the relationship between the curved Dirac matrices and the standard flat ones can now be used to compute $\gamma^\mu(x)\partial_\mu$ in equation (1), yielding

$$\gamma^\mu(x)\partial_\mu = \gamma_0\partial_0 + \frac{1}{ar^2}\sum_{i,j}[(\xi x_i^2 - r^2)\gamma_i\nabla_i + \xi x_ix_j\gamma_i\nabla_j] \quad (13)$$

Proceeding exactly similarly, the product $\gamma^\mu(x)\Gamma_\mu(x)$ appearing in equation (1) can be explicitly calculated from equations (6), (7), (11), and (12). The resulting expression simplifies to an exceedingly compact form:

$$\gamma^\mu(x)\Gamma_\mu(x) = -\frac{3\dot{a}}{2a}\gamma_0 - \frac{\xi}{ar^2}\mathbf{r}\cdot\boldsymbol{\gamma} \quad (14)$$

Expressions (13) and (14) can now be readily substituted in equation (1) which succinctly reads

$$\left\{\left(\frac{\partial}{\partial t}+\frac{3\dot{a}}{2a}\right)-\frac{1}{a}\boldsymbol{\alpha}\cdot\nabla-im\gamma_0+\frac{\xi}{ar^2}\left[\mathbf{r}\cdot\boldsymbol{\alpha}+\sum_{i,j}\left(x_i^2\alpha_i\nabla_i+x_ix_j\alpha_i\nabla_j\right)\right]\right\}\psi=0 \quad (15)$$

where $\alpha_i=\gamma_0\gamma_i$. Equation (15) is the required Dirac equation incorporating the spatial curvature of the Robertson–Walker metric.

It is worth recalling that the term independent of ξ is the usual Dirac equation for the spatially flat hypersurfaces, whereas the effect of curvature is contained in the term depending explicitly on ξ. It is immediately clear that in the limiting situation characterized by the vanishing of the curvature constant ($k\to 0 \Leftrightarrow \xi\to 0$), equation (15) reduces to a form which is precisely the same as that obtained by Barut and Duru for spatially flat hypersurfaces.

A few concluding remarks are particularly appropriate at this stage. The present formulation of the Dirac equation in spatially curved hypersurfaces in the context of the Robertson–Walker space-times is general enough to account for the realistic description of particles in realms where gravity effects can no longer be neglected. In doing so, however, we have sacrificed simplicity for the sake of rigor and generality. Furthermore, the question concerning the solutions to the full curvature-dependent equation [equation (15)] is replete with formidable difficulties due primarily to the presence of the curvature constant k. The degree of difficulty is further aggravated by the unknown function $a(t)$ which will ultimately admit of some suitable parametrization (and thereby injecting a fair amount of flexibility) depending on whether the universe is dominated by matter or radiation. In any case, the effect of the additional term (proportional to ξ) is expected to play a pivotal role in determining the plausibility of solutions to the Dirac equation in all its gravitational manifestations.

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